TABLES OF ORTHOGONAL POLYNOMIALS WHEN THE INDEPENDENT VARIABLE 'x' IS IN THE GEOMETRIC PROGRESSION:

$x_r = 2^{r-2}$; $x_1 = 0$; $x_r = 2^{r-1}$; $x_1 = 1$

BY DALJIT SINGH

Indian Agricultural Research Institute, New Delhi

TABLES for fitting orthogonal polynomials to data which is equispaced have been given by Fisher and Yates (1943) and Aitkin (1933). However, in certain biological and agronomic experiments the independent variable is in a geometric series. To meet this end the present set of tables has been worked out. The series with common ratio 2 has been given as such series are most widely used in biological research. While series with any other ratio could be considered and appropriate tables constructed, no simple relationship could be found to convert these polynomials in a geometric series from one common ratio to another. The present set of tables has been constructed from $n = 3$ to 8. For $n = 3$, 4 and 5 the polynomials up to 2nd, 3rd and 4th degree have been given while for $n = 6$, 7 and 8 polynomials up to the 3rd degree have been worked out. Two sets of tables are given—the first having $x_r = 2^{r-2}$; $x_1 = 0$ and the second $x_r = 2^{r-1}$; $x_1 = 1$.

The equation to be fitted is

$$y = b_0Z_0 + b_1Z_1 + b_2Z_2 + b_3Z_3 + \ldots \qquad (1)$$

where Z_p is polynomial of the p-th degree in x and $\Sigma Z_pZ_q = 0$; $p \neq q$; $Z_0 = 1$. The least square solutions of the Zs are given by the equation:

$$Z_p = \frac{1}{D^{(p-1)}} \begin{vmatrix} (0) & (1) \ldots (p) \\ (1) & (2) \ldots (p+1) \\ (p-1) & (p) \ldots (2p-1) \\ 1 & x \ldots x^p \end{vmatrix} \qquad (2)$$

where

$$D^{(p-1)} = \begin{vmatrix} (0) & (1) \ldots (p-1) \\ (1) & (2) \ldots (p) \\ (p-1) & (p) \ldots (2p-2) \end{vmatrix}$$

and $\quad (r) = \Sigma x^r$.

Equation (2) may be finally expanded in the form

$$Z_p = \Sigma\, C_{p,\, p-s}\, x^{p-s} \tag{3}$$

From (2) it is easy to see that the coefficient of x^p is $D^{(p-1)}/D^{(p-1)} = 1$. Therefore, $C_{p,\, p} = 1$.

Equation (2) may also be expressed as

$$Z_p = \frac{\left\{ \sum_{s=0}^{p} d_{p,\, p-s}\, x^{p-s} \right\}}{d_{p,\, p}} \tag{4}$$

where $d_{p,\, q}$ are relatively prime integers.

The *d*s are given in Table II (*a*).

Let

$$Z'' = \sum_{s=0}^{p} d_{p,\, p-s}\, x^{p-s} \tag{5}$$

i.e.,

$$Z = \frac{Z''}{d_{p,\, p}}. \quad \text{Also} \quad C_{p,\, p-s} = \frac{d_{p,\, p-s}}{d_{p,\, p}} \tag{6}$$

Giving values 0, 1, 2, 4, 8, etc., to x in (5) tables for Z'' may be constructed. If R is the highest common factor of the numbers Z'' and $RZ' = Z''$ then

$$Z = \frac{Z''}{d_{p,\, p}} = \frac{R}{d_{p,\, p}}\, Z' = \frac{Z'}{L} \tag{7}$$

where $L = d_{p,\, p}/R$.

Corresponding to (1) we have also the equation

$$y = b_0' Z_0' + b_1' Z_1' + b_2' Z_2' + b_3' Z_3' + \ldots \tag{8}$$

for which the least square solution of b' is given by $b' = \Sigma\, yZ'/\Sigma\, Z'^2$ corresponding to the solution $b = \Sigma\, yZ/\Sigma\, Z^2$ of (1). Also $b = Lb'$.

Table I (*a*) gives the values of Z' and other constants required for working out the b values corresponding to (1). The essential arrangement of each table is along the lines of Fisher and Yates' Tables. Against each x are given the numbers Z'; next the values of $\Sigma\, Z'^2$ followed by those of L. From these any b is found by the relation

$$b = \frac{L\, \Sigma\, yZ'}{\Sigma\, Z'^2} \tag{9}$$

The sum of squares due to fitting b is equal to $\{\Sigma yZ'\}^2/\Sigma Z'^2$. If S^2 is an estimate of the standard deviation the standard error of b is equal to $LS/\sqrt{Z'^2}$.

The computation of the tables of orthogonal polynomials when $x_1 = 1$ may be similarly done. However, a considerable amount of labour may be saved by adopting the following procedure which utilizes some of the determinants used in construction of tables when $x_1 = 0$. Let

$$W_r = \begin{vmatrix} (0) & (1) \ldots (r) \\ (1) & (2) \ldots (r+1) \\ \\ (r-1) & (r) \ldots (2r-1) \\ 1 & x \ldots \quad x^r \end{vmatrix} = \sum_{s=0}^{r} f_{r,r-s}\, x^{r-s} \tag{10}$$

It can be seen that

$$f_{r,t} = (-1)^{r-t} \begin{vmatrix} (0) & (1) \ldots (t-1) & (t+1) \ldots (r) \\ (1) & (2) \ldots (t) & (t+2) \ldots (r+1) \\ \\ (r-1) & (r) \ldots (t+r-2) & (t+r) \ldots (2r-1) \end{vmatrix} \tag{11}$$

Denote the determinant in equation (11) by

$$[(0) \quad (1) \ldots (t-1) \quad (t+1) \ldots (r)].$$

Then

$$f_{r,t} = (-1)^{r-t} \quad [(0) \quad (1) \ldots (t-1) \quad (t+1) \ldots (r)] \tag{12}$$

Now the value of (s), $s \neq 0$ when $x_1 = 0$ and the number of variates $= n$ is the same as the value of (s) with $x_1 = 1$ and the number of variates $= n - 1$. Let $(\dot{s})$, $\dot{W}_r$ and $\dot{f}$ be the functions with $x_1 = 1$ and the number of values $= n - 1$ corresponding to (s), W_r and f with $x_1 = 0$ and number of variates $= n$. Then we have

$$(0) - (\dot{0}) = 1 \tag{13}$$

$$\dot{W}_r = \sum_{s=0}^{r} \dot{f}_{r,r-s}\, x^{r-s} \tag{14}$$

$$\dot{f}_{r,t} = (-1)^{r-t} [(0)(1) \ldots (t-1)(t+1) \ldots (r)] \tag{15}$$

If a is any diagonal element of a determinant A we have the value of A given by

$$A = a\,|A_a| + \Sigma\, b\,|A_b| \tag{16}$$

where A_a is the minor of a and $|A_b|$ the co-factor of b—the summation extending over the remaining elements b in the column (or row) containing a.

If a is replaced by $\dot{a}$ while other elements are left undisturbed we have

$$\dot{A} = \dot{a}\,|A_a| + \Sigma b\,|A_b| \tag{17}$$

by virtue of the fact that the minor of $\dot{a}$ is the same as that of a. Hence from equations (16) and (17) we get

$$\dot{A} = A + (\dot{a} - a)\,A_a \tag{18}$$

Using (18) in (12) and (15) we get

$$\dot{f}_{r,t} = f_{r,t} + (-1)^{r-t}\{(\dot{0}) - (0)\}\ [(2)\,(3)\ldots(t)\ (t+2)\ldots(r+1)]. \tag{19}$$

Applying (13) and (14) we finally get

$$\dot{f}_{r,t} = f_{r,t} + (-1)^{r-t+1}\ [(2)\,(3)\ldots(t)\ (t+2)\ldots(r+1)]. \tag{20}$$

Knowing $f_{r,t}$ and the determinants of lower order not involving the element (0), it is possible to calculate the $\dot{f}$ coefficients of (15). Thereafter, the procedure is the same as that for the computation of orthogonal polynomials when $x_1 = 0$. Tables I (*b*) and II (*b*) are the tables corresponding to Tables I (*a*) and II (*a*) when $x_1 = 0$.

Example

In an agronomic experiment three doses of superphosphate, *viz.*, 16, 32 and 64 lb. were applied to berseem and the yield figures of fodder in lb. for totals of six plots were as follows:—

Doses of super (t)	Yield in lb. (y)
Control (No manure)	1,016
16 lb. ..	2,967
32 lb. ..	4,202
64 lb. ..	5,245

By dividing the doses by 16 we obtain the relation

$$x = \frac{t}{16}. \tag{21}$$

The polynomial to be fitted is

$$y = A_0 + A_1 t + A_2 t^2 + A_3 t^3 \tag{22}$$

while in terms of x the polynomial is

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3. \tag{23}$$

We now refer to Table I (*a*) for $n = 4$. The product-moments $\Sigma yz'$, b' and other quantities are given in the table below:

Degree of fitting	$\Sigma yZ'$	$b' = \frac{\Sigma yZ'}{\Sigma Z'^2}$	$L = 4$, $b = Lb' = 4b'$	SS due to $b = \frac{(\Sigma yZ')^2}{\Sigma Z'^2}$
0	1,3430	3357·5000,000	3357·5000,000	4509,1225·00
1	3,5394	252·8142,857	1011·2571,429	894,8108·83
2	−1,2147	−78·8766,234	−276·0681,818	95,8114·34
3	721	6·5545,455	30·0416,667	4725·83
		Total of SS due to b	..	5500,2174·00

Also $\Sigma y^2 = 5500,2174$. This agrees with the total of SS due to b.

Using equations (3) and (6) in (1) and by equating coefficients of like powers of x we get from (23)

$$a_0 = b_3 C_{3,0} + b_2 C_{2,0} + b_1 C_{1,0} + b_0 C_{0,0}$$

$$= b_3 \left(\frac{d_{3,0}}{d_{3,3}}\right) + b_2 \left(\frac{d_{2,0}}{d_{2,2}}\right) + b_1 \left(\frac{d_{1,0}}{d_{1,1}}\right) + b_0 \left(\frac{d_{0,0}}{d_{0,0}}\right) \tag{24}$$

$$a_1 = b_3 C_{3,1} + b_2 C_{2,1} + b_1 C_{1,1}$$

$$= b_3 \left(\frac{d_{3,1}}{d_{3,3}}\right) + b_2 \left(\frac{d_{2,1}}{d_{2,2}}\right) + b_1 \left(\frac{d_{1,1}}{d_{1,1}}\right) \tag{25}$$

$$a_2 = b_3 C_{3,2} + b_2 C_{2,2}$$

$$= b_3 \left(\frac{d_{3,2}}{d_{3,3}}\right) + b_2 \left(\frac{d_{2,2}}{d_{2,2}}\right) \tag{26}$$

$$a_3 = b_3 C_{3,3}$$
$$= b_3 \left(\frac{d_{3,3}}{d_{3,3}}\right) \qquad (27)$$

Substituting the values of b_3, b_2, b_1 and b_0 and the d values corresponding to $n = 4$ from Table II (a) in (24) we get

$$a_0 = (30\cdot0417)\left(\frac{-36}{55}\right) - (276\cdot0682)\left(\frac{14}{7}\right)$$
$$+ (1011\cdot2571)\left(\frac{-7}{4}\right) + (3357\cdot5000)\left(\frac{1}{1}\right)$$
$$= 1016\cdot0000.$$

Similarly we get

$$a_1 = 2369\cdot0833, \quad a_2 = -448\cdot1250, \quad a_3 = 30\cdot0417.$$

From (21), (22) and (23) we get

$$A_0 = a_0, \quad A_1 = \frac{a_1}{16}, \quad A_2 = \frac{a_2}{256}, \quad A_3 = \frac{a_3}{4096}. \qquad (28)$$

Using the values of a_0, a_1, etc., in (28) we get

$$A_0 = 1016\cdot0000, \quad A_1 = \frac{2369\cdot0833}{16} = 148\cdot0677,$$

$$A_2 = \frac{-448\cdot1250}{256} = -1\cdot7505, \quad A_3 = \frac{30\cdot0417}{4096} = 0\cdot007334.$$

Finally we have the relation

$$y = 1016\cdot0000 + 148\cdot0677\, t - 1\cdot7505\, t^2 + 0\cdot007334\, t^3.$$

Summary

In some biological, biochemical and agronomic experiments quantitative observations are taken on the effect of certain chemicals where the doses are given at intervals which proceed in geometric series of the type 2^r rather than in arithmetic progression. Two sets of tables for fitting orthogonal polynomials to such data have been constructed—the first including the control ($x_1 = 0$) and the second without the control ($x_1 = 1$). The range of the tables considered is from $n = 3$ to 8. For $n = 3$, 4 and 5 polynomial constants up to 2nd, 3rd and 4th degree and for $n = 6$, 7 and 8 polynomial constants up to the 3rd degree have been constructed. The arrangement of these polynomial constants is similar to that given by Fisher and Yates.

A method has been evolved which considerably decreases the computational work of constructing the second set of polynomials ($x_1 = 1$) by utilizing values obtained in the construction of the first set of polynomials ($x_1 = 0$).

ACKNOWLEDGEMENT

The author is thankful to Dr. P. N. Saxena and Shri M. N. Das for their helpful suggestions in the preparation of this paper.

REFERENCES

1. Aitken, A. C. .. "On the graduation of data by the orthogonal polynomials of least squares," *Proc. Roy. Soc. Edin.*, 1933, **53**, 54.

2. Fisher, R. A. and Yates, F. *Statistical Tables for Biological, Agricultural and Medical Research*, Oliver & Boyd, Edinburgh, 1943.

TABLE I (*a*)

Orthogonal Polynomials ($x_1 = 0$)

(3)

x	Z_1'	Z_2'
0	−1	+1
1	0	−2
2	+1	+1
$\Sigma Z'^2$	2	6
L	1	3

(4)

x	Z_1'	Z_2'	Z_3'
0	−7	+7	−3
1	−3	−4	+8
2	+1	−8	−6
4	+9	+5	+1
$\Sigma Z'^2$	140	154	110
L	4	7/2	55/12

(5)

x	Z_1'	Z_2'	Z_3'	Z_4'
0	−3	+31	− 837	+21
1	−2	+ 2	+ 752	−64
2	−1	−19	+ 916	+56
4	+1	−37	−1016	−14
8	+5	+23	+ 185	+ 1
$\Sigma Z'^2$	40	3224	317,1610	7870
L	1	4	2015/24	3935/672

(6)

x	Z_1'	Z_2'	Z_3'
0	−31	+ 93	−3,5433
1	−25	+ 40	+ 4600
2	−19	− 6	+2,8078
4	− 7	− 77	+3,5289
8	+17	−135	−3,9829
16	+65	+ 85	+ 7295
$\Sigma Z'^2$	6510	4,1664	49,4991,1360
L	6	7/2	1240/3

(7)

x	Z_1'	Z_2'	Z_3'
0	− 9	+1,1811	−180,7083
1	− 8	+ 7750	− 54,9568
2	− 7	+ 3955	+ 45,1768
4	− 5	− 2837	+175,7730
8	− 1	−1,3229	+208,5603
16	+ 7	−2,1245	−237,2543
32	+23	+1,3795	+ 43,4093
$\Sigma Z'^2$	798	10,3991,1306	17,0284,1992,7344
L	1	133	7,1827/24

(8)

x	Z_1'	Z_2'	Z_3'
0	−127	+3,2385	−141,8463
1	−119	+2,5788	− 81,4728
2	−111	+1,9408	− 27,2962
4	− 95	+ 7299	+ 63,3125
8	− 63	−1,4315	+179,4753
16	+ 1	−4,7127	+198,9697
32	+129	−7,1087	−234,2687
64	+385	+4,7649	+ 43,1265
$\Sigma Z'^2$	22,0472	118,9341,7158	16,0053,7887,8434
L	8	217/2	4231/12

TABLE I (*b*)

Orthogonal Polynomials ($x_1 = 1$)

(3)

x	Z_1'	Z_2'
1	−4	+2
2	−1	−3
4	+5	+1
$\Sigma Z'^2$	42	14
L	3	7/3

(4)

x	Z_1'	Z_2'	Z_3'
1	−11	+20	− 8
2	− 7	− 4	+14
4	+ 1	−29	− 7
8	+17	+13	+ 1
$\Sigma Z'^2$	460	1426	310
L	4	23/6	155/84

(5)

x	Z_1'	Z_2'	Z_3'	Z_4'
1	−26	+30	−176	+ 64
2	−21	+11	+ 76	−120
4	−11	−19	+252	+ 70
8	+ 9	−47	−181	− 15
16	+49	+25	+ 29	+ 1
$\Sigma Z'^2$	1319	4216	13,3858	2,3622
L	5	4/3	527/168	3937/3360

(6)

x	Z_1'	Z_2'	Z_3'
1	−19	+ 6696	−1,4632
2	−17	+ 4362	− 3194
4	−13	+ 163	+1,2594
8	− 5	− 6359	+2,0463
16	+11	−1,1899	−1,8317
32	+43	+ 7037	+ 3131
$\Sigma Z'^2$	2814	2,9543,2480	11,4582,4480
L	2	469/6	620/21

(7)

x	Z_1'	Z_2'	Z_3'
1	−120	+2994	−1,5168
2	−113	+2409	− 8536
4	− 99	+1297	+ 2678
8	− 71	− 695	+1,7577
16	− 15	−3751	+2,2627
32	+ 97	−6151	−2,3321
64	+321	+3897	+ 4143
$\Sigma Z'^2$	15,4688	8402,3962	16,9206,8752
L	7	29/3	671/168

(8)

x	Z_1'	Z_2'	Z_3'
1	−247	+23,6220	−294,7416
2	−239	+21,0192	−221,8422
4	−223	+15,9427	− 87,3689
8	−191	+ 6,3061	+138,0827
16	−127	−10,9015	+429,4619
32	+ 1	−37,0543	+489,7723
64	+257	−56,31 3	−554,2149
128	+769	+37,3761	+100,8507
$\Sigma Z'^2$	87,7880	7353,4288,2118	90,4426,1776,6930
L	8	1291/6	9035/84

TABLE II (*a*)

'*d*' *Coefficients in the Relation* $Z_r'' = \sum_{s=0}^{r} d_{r,\, r-s}\, x^{r-s}$

$(x_1=0)$

		(3)	(4)	(5)	(6)	(7)	(8)
Z_1''	$d_{1,1}$	1	4	1	6	1	8
	$d_{1,0}$	−1	−7	−3	−31	−9	−127
Z_2''	$d_{2,2}$	3	7	4	7	133	217
	$d_{2,1}$	−6	−29	−33	−113	−4194	−1,3411
	$d_{2,0}$	1	14	31	186	1,1811	6,4770
Z_3''	$d_{3,3}$	..	55	2015	2480	7,1827	4231
	$d_{3,2}$	..	−315	−2,3145	−5,7105	−328,9629	−38,4507
	$d_{3,1}$	..	392	5,9266	29,4823	3339,8162	762,5096
	$d_{3,0}$	..	−36	−2,0088	−21,2598	−4336,9992	−1702,1556
Z_4''	$d_{4,4}$	..	..	3935	..	..	..
	$d_{4,3}$	..	..	−5,3445	..	..	..
	$d_{4,2}$	..	..	20,1670	..	..	..
	$d_{4,1}$	..	..	−20,9280	..	..	..
	$d_{4,0}$	..	..	1,4112	..	..	..

TABLE II (*b*)

'*d*' *Coefficients in the Relation* $Z_r'' = \sum_{s=0}^{r} d_{r,\, r-s}\, x^{r-s}$

$(x_1=1)$

		(3)	(4)	(5)	(6)	(7)	(8)
Z_1''	$d_{1,1}$	3	4	5	2	7	8
	$d_{1,0}$	−7	−15	−31	−21	−127	−255
Z_2''	$d_{2,2}$	7	23	4	469	29	1291
	$d_{2,1}$	−36	−213	−69	−1,5411	−1842	−16,0041
	$d_{2,0}$	35	310	155	5,5118	1,0795	157,6070
Z_3''	$d_{3,3}$	..	155	527	1240	671	9035
	$d_{3,2}$	..	−1995	−1,2873	−5,8611	−6,2097	−164,8815
	$d_{3,1}$	..	6748	7,7266	64,7549	129,5770	6611,8696
	$d_{3,0}$	..	−5580	−9,4488	−120,4722	−378,2568	−3,1206,1860
Z_4''	$d_{4,4}$	..	..	3937	..	..	..
	$d_{4,3}$	..	..	−11,2995	..	..	..
	$d_{4,2}$	..	..	96,5650	..	..	..
	$d_{4,1}$	..	..	−278,3280	..	..	..
	$d_{4,0}$	..	..	214,1728	..	..	..